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String Interactions in $c = 1$ matrix model

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Abstract: We study string interactions in the fermionic formulation of the $c = 1$ matrix model. We give a precise nonperturbative description of the rolling tachyon state in the matrix model, and discuss S-matrix elements of the $c = 1$ string. As a first step to study string interactions, we compute the interaction of two decaying D0-branes in terms of free fermions. This computation is compared with the string theory cylinder diagram using the rolling tachyon ZZ boundary states.

Keywords: Bosonic Strings, Tachyon Condensation, Matrix Models.

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Contents

1. Introduction

It has been known for more than ten years that the non-critical $c = 1$ string can be described by the double scaling limit of a quantum mechanical matrix model, for reviews see e.g. [1, 2, 3]. In this duality the matrix model describes discretized world-sheets, which become continuum world-sheets in the double scaling limit. Recently, a new perspective on this duality has emerged which is in spirit closer the AdS/CFT duality and open/closed string duality. The crucial new insight obtained in [4, 5] and in [6] is that the matrix degrees of freedom of the quantum mechanical matrix model are to be identified with the open string tachyon degrees of freedom living on a system of unstable D0-branes in the $c = 1$ string theory. The full $c = 1$ string theory is recovered by taking a double scaling limit of the matrix model. For more discussion see e.g. the recent papers [7]–[12].

The $c = 1$ matrix quantum mechanics can be reformulated in terms of free fermions moving in an external potential. After taking the double scaling limit the potential becomes an upside-down quadratic potential, the system of free fermions becomes a non-relativistic $1 + 1$ dimensional field theory, and the ground state is a Fermi sea filled with an infinite number of fermions. The distance μ of the Fermi sea to the top of the potential can be identified with the inverse string coupling $1/g_s$ [2].

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It is an interesting question to ask how the string interactions and string diagrams are encoded in the free fermionic theory. In principle, one expects that different orders in string perturbation theory can be extracted from the matrix model by studying its large μ expansion. The tree-level scattering of closed string tachyons in the two dimensional string theory were computed in the matrix model (where one can compute them directly or by employing the free fermionic formulation) in [2, 16]. These scattering amplitudes were found to be in agreement with closed string computations when on the matrix model side external "legpole factors" are attached. These factors have poles at integer values of the external momenta, corresponding to discrete states in the string theory. After analytic continuation of the string amplitudes to Minkowski signature, these factors turn into a phase, which do not affect the physical scattering amplitudes.

However, the discrete states and the legpole factors play an important role in understanding the full structure of the $c = 1$ string theory. As shown in [3] gravitational effects are encoded in the legpole factors.

In the $c = 1$ matrix model the full perturbative S-matrix was also computed [3, 17]. Nonperturbatively this matrix model is unstable, since only one side of the potential is filled, and there are no closed-string states corresponding to the asymptotic region on the other side of the potential. A non-perturbatively unitary extension of the $c = 1$ matrix model is obtained by filling up both sides of the potential, in which case it does not describe the non-critical $c = 1$ string but the two dimensional 0A/0B string theory [10]. In this unitary theory an exact S-matrix can be computed [12, 17].

The partition function of the two dimensional string theory and its Kontsevich representation was worked out in [19].

Although many results have already been computed on the $c = 1$ matrix model side, in the present context we have a better understanding of the direct relation of the matrix model to the D0-branes of the two dimensional string theory. In particular, processes involving D-branes on the string side can now be studied with matrix model methods.

In [4, 5, 6] the decay of a single D-brane was analyzed in detail. In this picture, the decay of the D0-brane can be described by a single eigenvalue rolling down the potential. In the fermionic language one can describe this process by a single fermion moving from the top of the potential to asymptotic infinity, where it becomes relativistic. Bosonizing the relativistic fermion leads to the identification of the D-brane as a coherent state of tachyons in the asymptotic region. The closed string interactions are encoded in the interactions of the bosonized tachyon field. A general open-closed duality based on this description is suggested in [20].

In the string theory side, the decay of the unstable D-brane can be studied using a description involving boundary conformal theory. The relevant boundary states have two non-trivial pieces, one is the Liouville part which has been exactly determined in [13, 14], the other is the time-dependent piece which involves a timelike scalar and which was first studied in [15]. A precise determination of the time-dependent part of the boundary state is actually quite subtle and we will discuss it in more detail later in this paper.

As a first step in understanding in more detail the relation between the string worldsheet description and the dual matrix model, one would like to see how the string interactions are reproduced in the matrix model. In particular, since the matrix model has a description in terms of a simple system of free fermions in a quadratic potential, one would like to understand if and how string interactions between D-branes can be computed from the free fermionic theory.

So far, in various papers [4, 5, 6, 21] the disk one point function describing the decay of a single D-brane was computed and found to agree with the bosonized fermionic description. A further step is made in [9], where higher g_s corrections to the outgoing tachyon state were discussed using the S-matrix of the of the matrix model.

In the present paper we further analyze the question of string interactions in the presence of D-branes from the free fermionic point of view. The main point is to understand and compute string diagrams (the disk and the cylinder), including possible higher order and nonperturbative corrections, in terms of the free fermionic formulation of the matrix model.

In section 2 and 3 we develop the systematic formalism to compute string diagrams from the free fermionic point of view. The disk diagram, which has been computed before serves mainly as an example to develop our formalism. As described in section 3 the rolling tachyon can be viewed as a combination of in and out states. Here we use the S-matrix formalism to compute the disk amplitude and the string interactions between D0-branes. We conclude that the matrix model results are in agreement with the boundary state computations, if in the latter we take into account the exchange of closed string tachyons only. In particular, we do not see discrete modes contributing in the matrix model computation.

To compare the matrix model formulation of the interaction with the boundary state formulation, in section 4, we compute the tree level interaction between two decaying D0-branes. The Liouville part of the cylinder diagram is already computed in [14, 23]. Tensoring with a Dirichlet state in the time direction describes for example the cylinder diagram between two D-instantons [10, 11]. Here we consider the tree-level interaction between two decaying D0-branes. We will take as our boundary state the one that arises by an analytic continuation of the zero mode sector of the euclidean boundary state, and briefly discuss possible alternatives to this.

The comparison of the boundary state computation and matrix model results is further discussed in section 5.

2. The interaction of D-branes: a bosonic picture

In this section, we briefly review some background material pertaining to the $c = 1$ matrix model and the asymptotic bosonization picture of free fermions. Then we explain how to reproduce the brane "interaction" from this picture. The hamiltonian of free fermions in the inverted harmonic oscillators with Fermi energy μ is given by

$$H = \frac{1}{2} \int dx \left\{ \partial_x \psi^\dagger \partial_x \psi - x^2 \psi^\dagger \psi + 2\mu \psi^\dagger \psi \right\}. \tag{2.1}$$

We can introduce new left moving and right moving chiral fermions

$$\psi(x,t) = \frac{1}{\sqrt{2v(x)}} e^{+i\mu\tau - i\int^x dy v(y)} \psi_L(x,t) + \frac{1}{\sqrt{2v(x)}} e^{+i\mu\tau + i\int^x dy v(y)} \psi_R(x,t)\,, \tag{2.2}$$

where

$$v(x) = \sqrt{x^2 - 2\mu}\,. \tag{2.3}$$

The factors appearing in front of Ψ_L and Ψ_R are the WKB wavefunctions of the inverted harmonic oscillator. From now on, we will focus on the asymptotic region of large negative x, or the $q \equiv -\ln(-x) \to -\infty$ region. The reason for this will soon become clear. At this asymptotic infinity, the hamiltonian is reduced to the hamiltonian of relativistic fermions and we have

$$H = \frac{1}{2}\int dq\left[i\psi_R^\dagger \partial_q \psi_R - i\psi_L^\dagger \partial_q \psi_L + \mathcal{O}\left(\frac{1}{\mu}\right)\right]. \tag{2.4}$$

Here we suppressed $1/\mu$ corrections. This can be justified for the tree level computation in string theory. Possible systematic correction to this truncation will be discussed in section 3. The chiral fermions can be bosonized as

$$\begin{aligned}
\psi_R(q,t) &= \frac{1}{\sqrt{2\pi}} : \exp\left[\frac{i}{\sqrt{2}}\int^q (\Pi - \partial_q T)dq'\right] : \\
\psi_L(q,t) &= \frac{1}{\sqrt{2\pi}} : \exp\left[\frac{i}{\sqrt{2}}\int^q (\Pi + \partial_q T)dq'\right] : .
\end{aligned} \tag{2.5}$$

This scalar boson is, roughly speaking, a closed string tachyon modulo possible discrete modes. The hamiltonian of this scalar boson can be found easily by inserting (2.5) into (2.2) and evaluating (2.1). It has interactions of order $O(T^3)$ which are again suppressed by inverse powers of μ,

$$H = \frac{1}{2}\int dq\left(\Pi^2 + (\partial_q T)^2 + O(T^3)\right). \tag{2.6}$$

The field $T(q,t)$ satisfies the canonical equal time commutation relations

$$\left[T(q,t), \Pi(q',t)\right] = i\delta(q-q')\,. \tag{2.7}$$

Near asymptotic infinity, the bosonic field T can be expressed in terms of simple plane waves,

$$T(q,t) = \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{2k^2}}\left[T_k e^{-i|k|t+ikq} + T_k^\dagger e^{i|k|t-ikq}\right]. \tag{2.8}$$

The commutation relations of the creation and annihilation operators of this bosonic field are the usual ones,

$$[T_k, T_{k'}^\dagger] = |k|\delta(k-k')\,. \tag{2.9}$$

Including the normal ordering, (2.5) can now be written as

$$\begin{aligned}
\psi_R(q,t) &= \frac{1}{\sqrt{2\pi}} e^{-i\int_0^{+\infty}\frac{dk}{|k|}\bar{T}_k^\dagger e^{+ik(t-q)}} e^{-i\int_0^{+\infty}\frac{dk}{|k|}\bar{T}_k e^{-ik(t-q)}} \\
\psi_L(q,t) &= \frac{1}{\sqrt{2\pi}} e^{+i\int_{-\infty}^{0}\frac{dk}{|k|}T_k^\dagger e^{-ik(t+q)}} e^{+i\int_{-\infty}^{0}\frac{dk}{|k|}T_k e^{+ik(t+q)}} .
\end{aligned} \tag{2.10}$$

We distinguish T_k and $\bar{T}_k$, where the former acts naturally on the *out* vacuum and the latter naturally on the *in* vacuum. The relation between T_k and $\bar{T}_k$ can be found by comparing and equating the values of conserved quantities at very early and very late times. The relevant conserved quantities are related by an analytic continuation to the generators of the ground ring (which also correspond to conserved quantities). The result, as taken from [3], reads

$$\begin{aligned} \bar{T}_k^\dagger - \bar{T}_k = {} & 2^{ik} \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{i}{\mu}\right)^{n-1} \frac{\Gamma(1-ik)}{\Gamma(2-n-ik)} \int_{-\infty}^{0} dk_1 \cdots dk_n \times \\ & \times (T_{k_1}^\dagger - T_{k_1}) \cdots (T_{k_n}^\dagger - T_{k_n}) \delta(\pm|k_1| \cdots \pm |k_n| - k) \,. \end{aligned} \tag{2.11}$$

The leading one gives the tree level interaction, and since this expression includes all $1/\mu$ corrections, we can include higher order corrections [9].

States that describe the excitation of a single fermion out of the Fermi sea can be expressed as $\psi_R^\dagger(q,t)|\mu\rangle$ or $\psi_L^\dagger(q,t)|\mu\rangle$, where $|\mu\rangle$ denotes the Fermi sea. If we insert the classical trajectories of $x = -\hat{\lambda}\cosh t$ or $q = -\log\frac{\hat{\lambda}}{2} - \log\cosh t$ in the single fermion state, we get, for instance,

$$\psi_L^\dagger(q,t)|\mu\rangle = \frac{1}{\sqrt{2\pi}} e^{-i\int_0^\infty \frac{dk}{|k|} T_k^\dagger e^{-ik\log\hat{\lambda}}} |\mu\rangle \tag{2.12}$$

at asymptotic future infinity $t \to +\infty$. This single fermion state can be interpreted as outgoing coherent tachyon states emitted from the decaying D-brane [6].

The question is how we can compute the "interaction" from the free fermion theory. The strategy is as follows. Although the theory is free, there are non-vanishing multipoint functions of fermions, and we will interpret the two point function of fermions $\langle\mu|\psi(q,t)\psi^\dagger(q,t)|\mu\rangle$ as an "interaction amplitude" of D-branes. We should note that since each fermion in the two point function is interpreted as a decaying D-brane, to get a sensible S-matrix element of two interacting D-branes the (q,t) coordinates should be sent to asymptotic infinity as one usually does when constructing an S-matrix. Of course, this interpretation looks a bit different from standard quantum field theory at this moment, but in the next section we will see that once it is assembled together that the whole picture matches perfectly well with standard quantum field theory. At this asymptotic region, we can think about the "interaction" of two Fermi particles as

$$\begin{aligned} \langle\mu|\psi_L(q,t)\psi_L^\dagger(q',t')|\mu\rangle = {} & \langle\mu| \frac{1}{\sqrt{2\pi}} e^{-i\int_{-\infty}^0 \frac{dk}{|k|} T_k^\dagger e^{-ik(t+q)}} e^{-i\int_{-\infty}^0 \frac{dk}{|k|} T_k e^{+ik(t'+q')}} \times \\ & \times \frac{1}{\sqrt{2\pi}} e^{+i\int_{-\infty}^0 \frac{dk}{|k|} T_k^\dagger e^{-ik(t+q)}} e^{+i\int_{-\infty}^0 \frac{dk}{|k|} T_k e^{+ik(t'+q')}} |\mu\rangle \\ = {} & \frac{1}{2\pi} e^{+\int_0^{+\infty} \frac{dk}{k} e^{-ik((t+q)-(t'+q'))}} = \frac{1}{2\pi} e^{+\int_0^\infty \frac{dk}{k} e^{-ik(\log\hat{\lambda} - \log\hat{\lambda}')}} \end{aligned} \tag{2.13}$$

where in the last step we inserted the classical trajectory $q = -\log\frac{\hat{\lambda}}{2} - \log\cosh t$ in the correlation function of two Fermi fields at asymptotic infinity. We have also neglected the overlap of the wavefunctions of the fermions with the Fermi sea in the computation. The form of (2.13) suggests it is the ladder summation of cylinder diagrams. The single cylinder can be rewritten as

$$\mathcal{A}_{\text{cylinder}} \sim \int dk d\omega e^{-ik\log\hat{\lambda}} \frac{1}{k^2 - \omega^2} e^{+ik\log\hat{\lambda}'} \,. \tag{2.14}$$

We see that (2.14) is composed of three parts: a massless scalar particle propagator and two one point functions. The scalar particle is the closed string tachyon as we know from (2.5). We will discuss further the match in the section 4, but already at this level, it is apparent that even though (2.13) explains the closed string tachyon exchange perfectly, there is no signal of discrete modes.

3. The interaction of D-branes: a fermionic picture

We will now try to give a more precise definition of the rolling tachyon state in the $c = 1$ matrix model. Since the rolling tachyon has a very specific behavior at early and late times as we have considered in the previous section, we propose that in the Minkowski picture it is related to a combination of an in state and an out state. Bulk correlators in the $c = 1$ matrix model are evaluated between this in and out state. Both the in and out state have a free parameter, which defines the behavior at very early times $\sim a\exp(-X^0)$ and the behavior at very late times $\sim b\exp(X^0)$. We will therefore not view the parameters a, b as defining a single state in the matrix model, but as defining a pair of states consisting of an in state and an out state. The in state represents an incoming coherent wave of tachyons, the out state and outgoing coherent state of tachyons, which agrees with the intuitive picture of the rolling tachyon boundary state.

Of course, what we say here is basically the statement that a classical fermion trajectory is nothing but a D-brane, but so far in the literature the rolling tachyon has mainly been considered as single state in the matrix model, rather than as a combination of an in and an out state. However, if we want to compute S-matrix elements of the $c = 1$ string, we need to specify both the initial and final conditions of the D-brane with the rolling tachyon, and we will see below that this is indeed the right picture.

The map between closed string tachyon correlation functions and $c = 1$ quantities has been discussed in e.g. [2]. It involves the asymptotic bosonization described in the previous section, in particular the modes in (2.9) are the modes of the closed string tachyon. The same chiral fermions that appear asymptotically will also be used to construct the in and out states for the rolling tachyons, but whereas closed string tachyons correspond to fermion bilinears, for the rolling tachyon we will use a single fermion operator, as discussed in [9]. The relation between the exact $c = 1$ fermion and the asymptotic chiral fermions involves WKB factors [2] as in (2.2) that need to be removed in order to obtain a finite result. This removal of WKB factors is also exactly what is needed to obtain finite tachyon correlators [17].

The free fermion theory in the inverted harmonic oscillator has standard even and odd parity real delta-function normalized wavefunctions $\psi^{\pm}(\omega, x)$, as described in [17, 18]. We want to find the wavefunctions that have the appropriate plane wave behavior as $x \to -\infty$. The relevant wavefunctions are denoted $\chi_L^{\pm}(\omega, x)$; we could also have considered the other regime $x \to +\infty$ with corresponding wave functions $\chi_R(\omega, x)$ but will not do that here. This will be relevant once we are interested in the sinh boundary state (in the matrix model in which both sides of the potential are filled), for example. The $\chi_L^\pm$ are linear combinations of the standard wavefunctions, to be precise,

$$\begin{pmatrix} \chi_L^+(\omega,x) \\ \chi_L^-(\omega,x) \end{pmatrix} = \begin{pmatrix} s^-(\omega) & s^+(\omega) \\ s^+(\omega) & s^-(\omega) \end{pmatrix} \begin{pmatrix} \psi^+(\omega,x) \\ \psi^-(\omega,x) \end{pmatrix}. \tag{3.1}$$

with

$$s^\pm(\omega) = \sqrt{k(\omega)} \pm \frac{i}{\sqrt{k(\omega)}} \tag{3.2}$$

where

$$k(\omega) = \sqrt{1+e^{2\pi\omega}} - e^{\pi\omega}\,, \qquad k(\omega)^{-1} = \sqrt{1+e^{2\pi\omega}} + e^{\pi\omega}\,. \tag{3.3}$$

From (3.1) we readily obtain

$$\begin{pmatrix} \psi^+(\omega,x) \\ \psi^-(\omega,x) \end{pmatrix} = \frac{i}{4} \begin{pmatrix} s^-(\omega) & -s^+(\omega) \\ -s^+(\omega) & s^-(\omega) \end{pmatrix} \begin{pmatrix} \chi_L^+(\omega,x) \\ \chi_L^-(\omega,x) \end{pmatrix} \tag{3.4}$$

so that in terms of creation an annihilation operators (with standard commutation relations) we have

$$\psi(x,t) = \int d\omega\, e^{i\omega t} \left[\frac{i}{4} a_\omega^+ (s^-\chi_L^+ - s^+\chi_L^-) + \frac{i}{4} a_\omega^- (-s^+\chi_L^+ + s^-\chi_L^-) \right]. \tag{3.5}$$

The asymptotics of $\chi_L^\pm$ as $x \to -\infty$ are

$$\chi_L^\pm(\omega,x) \sim \frac{\mp 2i}{\left(2\pi|x|\sqrt{1+e^{2\pi\omega}}\right)^{1/2}} e^{\pm iF(|x|,\omega)} \tag{3.6}$$

with

$$F(x,\omega) = \frac{1}{4}|x|^2 - \omega \log|x| + \Phi(\omega) \tag{3.7}$$

with

$$\Phi(\omega) = \frac{\pi}{4} + \frac{i}{4} \log \left[\frac{\Gamma(\frac{1}{2} - i\omega)}{\Gamma(\frac{1}{2} + i\omega)} \right]. \tag{3.8}$$

We are interested, for the in-state, in the operator

$$\lim_{x\to-\infty} \psi^\dagger(t = -\log|x| + \log(\hat{\lambda}_{\text{in}}), x) \tag{3.9}$$

acting on the ground state $|\mu\rangle$. Similarly, for the out state, we are interested in the operator

$$\lim_{x\to-\infty} \psi(t = \log|x| - \log(\hat{\lambda}_{\text{out}}), x) \tag{3.10}$$

acting on the vacuum state $\langle\mu|$. To obtain a finite resulting expression, we need to restrict to χ_L^+ (the appropriate left/right moving part), and to remove the WKB piece

$$\frac{1}{\sqrt{|x|}} \exp\left(\frac{i}{4}|x|^2\right). \tag{3.11}$$

The resulting operators are most easily written in terms of the operators

$$\begin{aligned}
\Psi^{\dagger}_{\mathrm{in},\omega} &= \frac{e^{i\Phi(\omega)-i\omega\log\sqrt{\mu}}}{2(1+e^{2\pi\omega})^{1/4}}(s^{-}_{\omega}(a^{+}_{\omega})^{\dagger} - s^{+}_{\omega}(a^{-}_{\omega})^{\dagger}) \\
\Psi_{\mathrm{out},\omega} &= \frac{e^{i\Phi(\omega)-i\omega\log\sqrt{\mu}}}{2(1+e^{2\pi\omega})^{1/4}}(s^{-}_{\omega}a^{+}_{\omega} - s^{+}_{\omega}a^{-}_{\omega})
\end{aligned} \tag{3.12}$$

which will play a crucial role in what follows. They describe the modes of the asymptotic chiral fermions, but in terms of the *exact* annihilation and creation operators of the $c=1$ theory. The factors of $\log\sqrt{\mu}$ are put in by hand here in order to facilitate the comparison with world-sheet results later. The anti-commutation of the fermions are

$$\begin{aligned}
\{\Psi_{\mathrm{out},\omega}, \Psi^{\dagger}_{\mathrm{out},\omega'}\} &= \delta(\omega-\omega') \\
\{\Psi_{\mathrm{in},\omega}, \Psi^{\dagger}_{\mathrm{in},\omega'}\} &= \delta(\omega-\omega') \\
\{\Psi^{\dagger}_{\mathrm{in},\omega}, \Psi_{\mathrm{out},\omega'}\} &= -\frac{e^{\pi\omega}e^{2i\Phi(\omega)-i\omega\log\mu}}{\sqrt{1+e^{2\pi\omega}}}\delta(\omega-\omega') \\
\{\Psi_{\mathrm{in},\omega}, \Psi^{\dagger}_{\mathrm{out},\omega'}\} &= -\frac{e^{\pi\omega}e^{-2i\Phi(\omega)+i\omega\log\mu}}{\sqrt{1+e^{2\pi\omega}}}\delta(\omega-\omega')\,.
\end{aligned} \tag{3.13}$$

The factor that appears in the the commutation relations between the in and out modes is the reflection coefficient which is the S-matrix of the free fermion theory. Most of the non-trivial physics of the $c=1$ theory is contained in this reflection coefficient. It can be rewritten as

$$\begin{aligned}
R_{\omega} &= -\frac{e^{\pi\omega}e^{2i\Phi(\omega)-i\omega\log\mu}}{\sqrt{1+e^{2\pi\omega}}} \\
&= -\frac{ie^{\pi\omega-i\omega\log\mu}}{\sqrt{1+e^{2\pi\omega}}}\sqrt{\frac{\Gamma(\frac{1}{2}+i\omega)}{\Gamma(\frac{1}{2}-i\omega)}} \\
&= -\frac{i}{\sqrt{2\pi}}e^{\pi\omega/2-i\omega\log\mu}\Gamma(1/2+i\omega)
\end{aligned} \tag{3.14}$$

in view of the relation

$$\Gamma\left(\frac{1}{2}+i\omega\right)\Gamma\left(\frac{1}{2}-i\omega\right) = \frac{\pi}{\cosh\pi\omega}\,. \tag{3.15}$$

The factor R_{ω} decays exponentially at large negative ω, which guarantees that the results we obtain will be finite. This exponential decay indicates that only non-perturbative effects will access the region where the energy is negative.

The reflection coefficient has nonperturbative pieces and a perturbative expansion in $1/\mu$. The perturbative expansion is of the form

$$R_{\mu+\omega} = -ie^{-i\mu}\left[1+\frac{i}{2\mu}\left(\omega^2+\frac{1}{12}\right)+\mathcal{O}(\mu^{-2})\right] \tag{3.16}$$

with an irrelevant prefactor $-ie^{-i\mu}$.

Based on our previous discussion, we now propose that the operator that creates a D-brane with a rolling tachyon that as $x \to -\infty$ behaves as

$$T \sim \hat{\lambda}_{\rm in} e^{-t} \tag{3.17}$$

is

$$|{\rm in}\rangle \equiv D^{\dagger}_{{\rm in},\hat{\lambda}_{\rm in}} |\mu\rangle = \int d\omega\, e^{-i\omega \log \hat{\lambda}_{\rm in}} \Psi^{\dagger}_{{\rm in},\omega} |\mu\rangle \tag{3.18}$$

and similarly that the out operator that annihilates a D-brane with a rolling tachyon that as $x \to +\infty$ behaves as

$$T \sim \hat{\lambda}_{\rm out} e^{t} \tag{3.19}$$

is

$$\langle{\rm out}| \equiv \langle\mu| D_{{\rm out},\hat{\lambda}_{\rm out}} = \langle\mu| \int d\omega\, e^{-i\omega \log \hat{\lambda}_{\rm out}} \Psi_{{\rm out},\omega}\,. \tag{3.20}$$

Similarly, we can make in and out states that contain multiple branes by simply acting with the corresponding brane creation and annihilation operators.

Before continuing the discussion, we first test this proposal by computing the energy associated to the rolling tachyon. In string theory, this should be the one point function of the relevant ground ring operator in the rolling tachyon boundary state. For the present point of view the relevant calculation is easy to write down, since the energy of $(a^{\pm}_{\omega})^{\dagger}$ is ω. The exact expression for the energy therefore is given by computing the expectation value of ω between the in and the out state, which after normalization leads to the expression

$$E = \frac{\int_{-\infty}^{\mu} e^{-i\omega(\log \hat{\lambda}_{\rm in} + \log \hat{\lambda}_{\rm out})} \omega \frac{e^{\pi\omega}}{\sqrt{1+e^{2\pi\omega}}} \sqrt{\frac{\Gamma(\frac{1}{2}+i\omega)}{\Gamma(\frac{1}{2}-i\omega)}}}{\int_{-\infty}^{\mu} e^{-i\omega(\log \hat{\lambda}_{\rm in} + \log \hat{\lambda}_{\rm out})} \frac{e^{\pi\omega}}{\sqrt{1+e^{2\pi\omega}}} \sqrt{\frac{\Gamma(\frac{1}{2}+i\omega)}{\Gamma(\frac{1}{2}-i\omega)}}}\,. \tag{3.21}$$

In this expression we absorbed a factor of $\sqrt{\mu}$ in both $\hat{\lambda}_{\rm in}$ and $\hat{\lambda}_{\rm out}$.

Let us try to work out this expression. In the region where ω is close to μ, the factor $e^{\pi\omega}/\sqrt{1+e^{2\pi\omega}}$ can be dropped as it is equal to 1 plus nonperturbative corrections. We are left with

$$E = \frac{\int_{-\infty}^{\mu} e^{-i\omega(\log \hat{\lambda}_{\rm in} + \log \hat{\lambda}_{\rm out})} \omega \sqrt{\frac{\Gamma(\frac{1}{2}+\omega)}{\Gamma(\frac{1}{2}-i\omega)}}}{\int_{-\infty}^{\mu} e^{-i\omega(\log \hat{\lambda}_{\rm in} + \log \hat{\lambda}_{\rm out})} \sqrt{\frac{\Gamma(\frac{1}{2}+i\omega)}{\Gamma(\frac{1}{2}-i\omega)}}}\,. \tag{3.22}$$

If we assume the rolling eigenvalue has small overlap with the Fermi sea, we can extend the integral to the range from $-\infty$ to ∞, and since it is rapidly fluctuating we can use the method of stationary phase. The fluctuating determinant is subleading in the genus expansion, so we are interested in the leading stationary phase approximation. Using Stirling's formula, the gamma functions behave as

$$\sqrt{\frac{\Gamma(\frac{1}{2}+i\omega)}{\Gamma(\frac{1}{2}-i\omega)}} \sim \exp\left(i(\omega \log \omega - \omega) + \mathcal{O}\left(\frac{1}{\omega}\right)\right)\,. \tag{3.23}$$

The stationary phase is at $\omega = \hat{\lambda}_{\rm in}\hat{\lambda}_{\rm out}$ which is indeed where we expect the energy to be located, and the expression for the energy becomes

$$E = \frac{\partial}{\partial(-i\log(\hat{\lambda}_{\rm in}\hat{\lambda}_{\rm out}))}\log(\exp(-i\hat{\lambda}_{\rm in}\hat{\lambda}_{\rm out})) = \hat{\lambda}_{\rm in}\hat{\lambda}_{\rm out}\,. \tag{3.24}$$

This is indeed the correct energy for an orbit of the type

$$\lambda(x) \sim \hat{\lambda}_{\rm in}e^{-t} + \hat{\lambda}_{\rm out}e^{t}\,. \tag{3.25}$$

We can easily take the fluctuation determinant into account in this discussion. This merely adds an extra term $i/2$ to the energy, from which we see that this is indeed a subleading effect, since $\hat{\lambda}_{\rm in}$ and $\hat{\lambda}_{\rm out}$ both contain a factor of $\sqrt{\mu}$ that we absorbed at the beginning of this calculation.

3.1 Adding tachyons

The construction of the tachyon operators uses crucially an asymptotic bosonization procedure. To add tachyons to the game, we need the normal ordered products

$$\rho(x,t) =: \psi^\dagger(x,t)\psi(x,t) : \tag{3.26}$$

in the same in and out limits as above. This operator in boson language will become $\partial\phi$ or $\bar{\partial}\phi$. Thus we can read off the tachyon operators by looking at modes of $\rho(x,t)$. The normal ordering is with respect to the Fermi surface, but this is only relevant for the zero energy tachyon which we will not consider.

It is straightforward to write down the tachyon operators from the normal ordered product of two fermions. We have the operators

$$\begin{aligned} T^\dagger_{{\rm in},k} &= \int d\omega\,\Psi_{{\rm in},\omega+k}\Psi^\dagger_{{\rm in},\omega} \\ T_{{\rm out},k} &= \int d\omega\,\Psi_{{\rm out},\omega-k}\Psi^\dagger_{{\rm out},\omega}\,. \end{aligned} \tag{3.27}$$

These have the property that for $k>0$, $T^\dagger_{{\rm in},k}$ creates a positive energy incoming tachyon of momentum k, whereas $T_{{\rm out},k}$ acting on $\langle\mu|$ creates an out state with a positive energy outgoing tachyon of momentum k. Clearly, $T^\dagger_{{\rm in},k} = T_{{\rm in},-k}$ and similarly for $T_{\rm out}$. The commutation relations are indeed those of a free scalar

$$[T_{{\rm in},k}, T^\dagger_{{\rm in},k'}] = [T_{{\rm out},k}, T^\dagger_{{\rm out},k'}] = k\delta(k-k')\,. \tag{3.28}$$

Up to a rescaling, these are of course the same as the tachyon modes that we wrote in (2.8).

We now have defined brane creation and annihilation operators, and tachyon creation and annihilation operators. We can now write down a general amplitude involving branes (with prescribed open string tachyon profiles) and closed string tachyons, and the matrix model result for such an amplitude can simply be worked out used the commutation relations of the $\Psi_{\rm in,out}$ operators, and the fact that

$$\begin{aligned} \Psi^\dagger_{{\rm in},\omega}|\mu\rangle &= \Psi^\dagger_{{\rm out},\omega}|\mu\rangle = 0 \qquad \text{for } \omega > \mu \\ \Psi_{{\rm in},\omega}|\mu\rangle &= \Psi_{{\rm out},\omega}|\mu\rangle = 0 \qquad \text{for } \omega < \mu\,. \end{aligned}$$

Notice that it matters whether we first create a brane and then a tachyon, or whether we first create a tachyon and then a brane,

$$[D^\dagger_{\mathrm{in},\lambda}, T^\dagger_{\mathrm{in},k}] = e^{-ik\log\hat\lambda_{\mathrm{in}}} D^\dagger_{\mathrm{in},\lambda}\,. \tag{3.29}$$

This is a signal of the fact that there is duality between bosons and fermions (bosonization), or equivalently, that there is a duality between open and closed strings. The complete Hilbert space can be spanned either by closed strings states plus their solitons (the closed string = boson picture), or by systems of branes and anti-branes (the open string = fermion picture).

To illustrate this picture, we will now compute a few quantities and check the results against results in the literature and our cylinder calculation described in section (4).

Tachyon correlators have been computed in e.g. [17] with precisely these conventions. The world-sheet tachyons are obtained by including a leg-pole factor, e.g.

$$T^\dagger_{\mathrm{in},\omega;\mathrm{world-sheet}} = \frac{\Gamma(i\omega)}{\Gamma(-i\omega)} T^\dagger_{\mathrm{in},\omega}\,. \tag{3.30}$$

In the conventions of [3] we should include an extra factor of $\sqrt{2\pi}$ in the right hand side of this expression. One also has the option of including a factor $1/\mu$ on the right hand side, this is a simple choice of normalization, and one could also include phases in the definition but we will not do that here.

The first quantity we will consider is the emission probability of a tachyon by a rolling tachyon. For this purpose we need to compute

$$\frac{\left\langle \mu | D_{\mathrm{out},\hat\lambda_{\mathrm{out}}} T_{\mathrm{out},k} D^\dagger_{\mathrm{in},\hat\lambda_{\mathrm{in}}} | \mu \right\rangle}{\left\langle \mu | D_{\mathrm{out},\hat\lambda_{\mathrm{out}}} D^\dagger_{\mathrm{in},\hat\lambda_{\mathrm{in}}} | \mu \right\rangle}\,. \tag{3.31}$$

This can easily be computed using Wick contractions, and we find

$$-\frac{\int_{-\infty}^{\min(\mu,\mu+k)} d\omega e^{-i\omega\log\hat\lambda_{\mathrm{out}} - i(\omega-k)\log\hat\lambda_{\mathrm{in}}} R_{\omega-k}}{\int_{-\infty}^{\mu} d\omega e^{-i\omega\log\hat\lambda_{\mathrm{out}} - i\omega\log\hat\lambda_{\mathrm{in}}} R_{\omega}}\,. \tag{3.32}$$

If we compute numerator and denominator in the stationary phase approximation, which is the leading contribution in the $1/\mu$ expansion as discussed above, we find for the emission probability

$$\mathcal{A} = -e^{-ik\log\hat\lambda_{\mathrm{out}}}\,. \tag{3.33}$$

Up to a prefactor which depends on the choice of convention, this is the answer one expects [6] for the emission probability for a tachyon from the decaying brane. Inserting $T^\dagger_{\mathrm{out}}$ would naturally describe the absorption of the tachyon by a decaying brane. We could also have chosen the order of the out brane and out tachyon operator the other way around. Then the results of the calculation would have no tree level contribution.

A similar calculation yields for the tree-level contribution to the emission amplitude

$$\frac{\left\langle \mu | D_{\mathrm{out},\hat\lambda_{\mathrm{out}}} T^\dagger_{\mathrm{in},k} D^\dagger_{\mathrm{in},\hat\lambda_{\mathrm{in}}} | \mu \right\rangle}{\left\langle \mu | D_{\mathrm{out},\hat\lambda_{\mathrm{out}}} D^\dagger_{\mathrm{in},\hat\lambda_{\mathrm{in}}} | \mu \right\rangle} = -e^{-ik\log\hat\lambda_{\mathrm{in}}}\,. \tag{3.34}$$

Inserting T_{in} again would describe absorption of a tachyon by an incoming brane.

3.2 Two branes

Finally, we consider the situation with two D-branes. From the boundary state point of view, we have one in-boundary state corresponding to one D-brane, and another out-boundary state corresponding to the complex conjugated D-brane. Therefore, we propose that the relevant matrix model calculation that describes the interaction between two D-branes is

$$\mathcal{A} = \left\langle D_{\mathrm{out},\hat{\lambda}^1_{\mathrm{out}}} D_{\mathrm{in},\hat{\lambda}^2_{\mathrm{in}}} D^\dagger_{\mathrm{out},\hat{\lambda}^2_{\mathrm{out}}} D^\dagger_{\mathrm{in},\hat{\lambda}^1_{\mathrm{in}}} \right\rangle \tag{3.35}$$

The operators $D_{\mathrm{in},\hat{\lambda}^2_{\mathrm{in}}}$ and $D^\dagger_{\mathrm{out},\hat{\lambda}^2_{\mathrm{out}}}$ that describe the second brane have been complex conjugated, so that the role of in and out has been interchanged, and their definition therefore involves phases $e^{i\omega \log \hat{\lambda}^2_{\mathrm{in}}}$ and $e^{i\omega \log \hat{\lambda}^2_{\mathrm{out}}}$.

The amplitude in question can be rewritten using Wick's theorem as

$$\begin{aligned}\mathcal{A} = &\int_{-\infty}^{\mu} d\omega R^*_\omega e^{i\omega(\log \hat{\lambda}^2_{\mathrm{in}} + \log \hat{\lambda}^2_{\mathrm{out}})} \int_{-\infty}^{\mu} d\omega R_\omega e^{-i\omega(\log \hat{\lambda}^1_{\mathrm{in}} + \log \hat{\lambda}^1_{\mathrm{out}})} - \\ &- \int_{-\infty}^{\mu} d\omega e^{i\omega(\log \hat{\lambda}^2_{\mathrm{in}} - \log \hat{\lambda}^1_{\mathrm{in}})} \int_{-\infty}^{\mu} d\omega e^{i\omega(\log \hat{\lambda}^2_{\mathrm{out}} - \log \hat{\lambda}^1_{\mathrm{out}})} .\end{aligned} \tag{3.36}$$

The first term in the expression has the interpretation as interactions between the incoming and outgoing branes, the second term gives the interaction between the incoming-incoming and outgoing-outgoing branes. More precisely, in terms of the boundary state computation, each integral corresponds to the interaction between the relevant incoming/outgoing states. The incoming/outgoing states are described by choosing the relevant time contour, or picking the part of the full brane computation. Thus the above four-point calculation gives the "square" of the cylinder. Also notice that the above result contains full information of the interaction beyond cylinder diagram. The information is contained in the reflection coefficient R_ω.

To compare with the string theory calculation therefore we take a single integral piece. Focusing on the piece in the second term which describes the interaction of two incoming branes, we have

$$\int_{-\infty}^{\mu} d\omega e^{i\omega(\log \hat{\lambda}^2_{\mathrm{in}} - \log \hat{\lambda}^1_{\mathrm{in}})} \tag{3.37}$$

which can be evaluated to be

$$e^{i\mu(\log \hat{\lambda}^2_{\mathrm{in}} - \log \hat{\lambda}^1_{\mathrm{in}})} \left(\pi\delta(\log \hat{\lambda}^2_{\mathrm{in}} - \log \hat{\lambda}^1_{\mathrm{in}}) - \frac{i}{\log \hat{\lambda}^2_{\mathrm{in}} - \log \hat{\lambda}^1_{\mathrm{in}}} \right) . \tag{3.38}$$

The presence of the δ-function is understood from the exclusion principle, one cannot place two fermions on the same location. Assuming $\hat{\lambda}_1$ is not equal $\hat{\lambda}_2$, and neglecting overlap with the Fermi-sea we get for the cylinder diagram between two incoming branes[1]

$$\exp\left(-\gamma_{\mathrm{E}} + \int_0^\infty \frac{dk}{k} e^{ik(\log \hat{\lambda}^2_{\mathrm{in}} - \log \hat{\lambda}^1_{\mathrm{in}})} \right) . \tag{3.39}$$

[1] Note that the integration is divergent when k approaches zero. We regulate the divergence, and Euler's constant γ_{E} appears by the process of regularization.

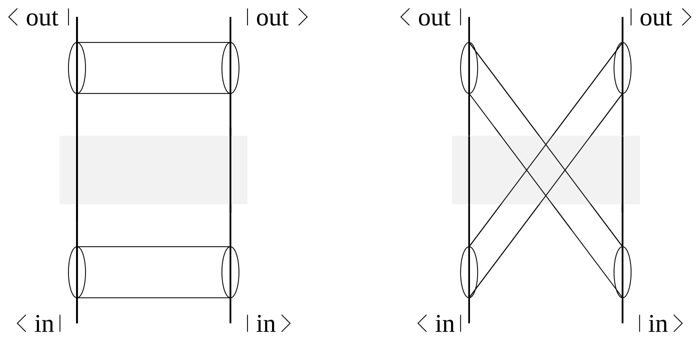

Figure 1: Interactions of two D0-branes are composed of two terms each of which contain two cylinders. The shaded area denotes the strongly coupled region.

This result has the interpretation as the exponential of a single string theory annulus diagram, in agreement with the computations in section 2. Note that the full computation of (3.36) has the other piece multiplied to (3.39), which has the interpretation as the annulus diagram between outgoing states. Thus (3.36) can be thought of as the combination of two annulus diagrams as depicted in figure 1. The two point function calculated in section 2 corresponds to the contribution of one of the two cylinders. We see that the cylinders are originated only from the asymptotic infinities. This is because these cylinders are the genus suppressed ones as we explain in appendix A. Closed string modes emitted or absorbed at the strongly coupled region will be affected by higher genus corrections.

Notice that (3.36) is the exact answer in the matrix model, and that again we see no sign of divergences and/or the exchange of discrete states.

4. Annulus diagram from boundary states

4.1 Preliminaries

In this section, we consider the interaction of two unstable D0-branes in terms of boundary state formalism, and compare with the results of the previous sections. Each brane carries an open string tachyonic degree of freedom. The rolling of the tachyon can be described by boundary conformal field theory with a boundary interaction

$$S = \frac{\lambda}{2}\int dt e^{X^0(t)} \,. \tag{4.1}$$

This interaction describes a decaying (half) brane solution, where the tachyon rolls down its potential. One can also consider a bounce (full brane) solution with a cosh boundary interaction

$$S = \lambda \int dt \cosh X^0(t) \,. \tag{4.2}$$

This perturbation can be interpreted as a brane building up from a finely tuned incoming coherent tachyon state, and decaying again by rolling down the potential.

In the matrix model, the decaying solution is interpreted as an eigenvalue rolling down the potential, while the bouncing solution as an eigenvalue coming in from asymptotic infinity, rolling up the potential, then rolling down again. The precise interpretation of these boundary states in terms of free fermions was discussed in detail in section 3.

The boundary state in the two dimensional string theory decomposes into an X^0 dependent (rolling tachyon) part, a Liouville part, and a ghost part

$$|B\rangle = |B\rangle_{X^0} \otimes |B\rangle_L \otimes |B\rangle_{\text{ghost}} . \tag{4.3}$$

The unstable D0-branes are described by the ZZ boundary states in the Liouville theory [13, 14]. The rolling part of the boundary state is less well understood, and we first turn to a brief discussion of the rolling part of the boundary state.

4.2 The boundary state for the rolling tachyon

We are interested in determining the boundary state that describes the boundary interaction (4.1) in the free theory that consists of a timelike scalar. This is somewhat subtle issue. Two alternative representations of this boundary state have been proposed in the literature. In the first one we expand the boundary state in Ishibashi states $|\omega\rangle\rangle$ built out of normalizable operators in the free timelike scalar theory. This representation is naturally obtained by considering a spacelike Liouville theory with arbitrary background charge, and by taking a limit where the background charge vanishes. This leads to the boundary state [22]

$$|B\rangle_{X^0} = \int d\omega e^{-i\omega \log \hat{\lambda}} \frac{-i\pi}{\sinh \pi\omega} |\omega\rangle\rangle , \tag{4.4}$$

where

$$\hat{\lambda} = \pi\lambda . \tag{4.5}$$

There is an alternative representation of the boundary state, obtained by Wick rotating the boundary state for an euclidean boson with boundary interaction e^{iX}; this boundary state is an expansion in Ishibashi states built on non-normalizable operators and its form can be obtained from [25, 26]

$$|B\rangle_{X^0} = \sum_j \sum_{m\geq 0}^{j} \binom{j+m}{2m} (i\hat{\lambda})^{2m} |j, m, m\rangle\rangle . \tag{4.6}$$

Here the sum is over nonnegative half-integer j, and the sum over m is such that $m + j$ is an integer. For some more discussion of these boundary states see e.g. [27] and references therein. This representation of the boundary state was used in the original work of Sen [15].

Both representations (4.4) and (4.6) are obtained by indirect procedures, either taking a limit of Liouville theory or by a suitable analytic continuation. One can also try to directly compute the boundary state by considering disk diagrams with an operator inserted in the middle of the disk, see e.g. [8, 22, 28]. These calculations are in principle well-defined, except for the contribution of the zero-mode, which is where all the subtleties arise. If we represent the operator inserted in the middle of the disk through the state-operator correspondence as

$$V = e^{i\omega X^0} \mathcal{P}(\partial X^0, \partial^2 X^0, \dots, \bar{\partial} X^0, \bar{\partial}^2 X^0, \dots) \tag{4.7}$$

for some polynomial $\mathcal{P}$, then the disk calculation will lead to an answer of the form (where t is the zero mode of X^0)

$$\int dte^{i\omega t}\sum_{n=0}^{\infty}\hat{\lambda}^n e^{nt}c_n(\mathcal{P}) \qquad (4.8)$$

where $c_n(\mathcal{P})$ does not depend on t or $\hat{\lambda}$. It represents the disk calculation with n insertions of the boundary operator and a zero momentum operator $\mathcal{P}(\partial X^0, \bar{\partial}X^0, \ldots)$ inserted in the middle of the disk. We can now see how we have to treat (4.8) in order to recover either (4.4) or (4.6).

To recover (4.6), we should apply momentum conservation to (4.8), and interpret the t integral as

$$\int e^{i\omega t}e^{nt}dt \sim \delta(\omega - in) \qquad (4.9)$$

which one would naturally get by Wick rotating the zero mode and viewing the integral as a Fourier integral. With this interpretation, there is only a contribution for imaginary momenta $\omega = in$ and this leads to the boundary state (4.6). The coefficients $c_n(\mathcal{P})$ have been computed for various $\mathcal{P}$ in [8], leading for example to the results

$$c_n(1) = (-1)^n\,, \qquad c_n(\partial X^0, \bar{\partial}X^0) \sim (-1)^n - 2\delta_{n,0}\,. \qquad (4.10)$$

The $\delta_{n,0}$ that appears is directly related to the nontrivial combination of Ishibashi states present in (4.6).

To recover (4.4), we first consider $\mathcal{P} = 1$. Then (4.8) becomes

$$\int dte^{i\omega t}\rho(\lambda, t) \qquad (4.11)$$

with

$$\rho(\lambda, t) = \frac{1}{1 + \hat{\lambda}e^t}\,. \qquad (4.12)$$

In order to make sense of (4.11), we have to choose an integration prescription. In [21] the integral is defined by taking the contour over the real axis with a small rotation $t \to t(1-i\epsilon)$, to include the poles of $\rho(t)$. With this definition we find

$$\int dte^{i\omega t}\rho(\lambda, t) = e^{-i\omega\log\hat{\lambda}}\frac{-i\pi}{\sinh\pi\omega} \qquad (4.13)$$

which is precisely what we have in (4.4). However, we only considered the case $\mathcal{P} = 1$, and we should extend this calculation to include all $\mathcal{P}$. One can see from simple examples that this is not at all straightforward. For example, taking $\mathcal{P} = \partial^2 X^0\bar{\partial}^2 X^0$ [8] leads in (4.8) to

$$\int dte^{i\omega t}(\rho(\lambda, t) - 2 + \hat{\lambda}e^t) \qquad (4.14)$$

and the contour integral given above does not yet give a good definition of the integral $\int dte^{i\omega t}e^t$. One way to define these integrals is to demand that the resulting boundary state that we obtain is conformally invariant, i.e. obeys

$$(L_n - \bar{L}_{-n})|B\rangle_{X^0} = 0\,. \qquad (4.15)$$

Since the highest weight representations of the Virasoro algebra with $c = 1$ and conformal weight $-\omega^2 < 0$ do not contain any null vectors,[2,3] it is sufficient to find the disk diagrams with $\mathcal{P} = 1$ only; all other disk diagrams then follow from (4.15). Thus, conformal invariance of the boundary state plus the prescription (4.13) provide a unique answer for all ill-defined integrals such as (4.14), and the resulting answer is then given by (4.4). All this is true except at zero momentum. For $\omega = 0$, the corresponding highest weight representation has null vectors and (4.15) is not sufficient to determine the boundary state. This will not play an important role in what follows, but we note that the contribution to (4.6) with zero momentum is the sum of the terms with $m = 0$, so it simply is $\sum_j |j, 0, 0\rangle\rangle$, which behaves as if there were no null vectors anyway.

We have now motivated the two representations (4.4) and (4.6) from direct world-sheet calculations. It appears that some of the information of (4.6) got lost in relating it to (4.4); in particular, we could have taken the zero mode approximation of (4.6) as in [15], and from that we could also have derived (4.4) via a Fourier transformation. On the other hand, the zero mode structure of (4.6) plus conformal invariance determine (4.4) up to the contribution of $\omega = 0$, so perhaps it should be possible to somehow re-obtain (4.6) from (4.4) as well.

In this paper we are mainly interested in computing an annulus diagram involving two boundary states. Though (4.4) and (4.6) are related, it is not clear whether and how calculations involving (4.4) are identical to calculations involving (4.6). If we use (4.6) one might be inclined to write down the following expression for the annulus amplitude

$$\mathcal{A} \sim \sum_j \sum_{m \geq 0}^{j} (\hat{\lambda}_1 \hat{\lambda}_2)^{2m} \binom{j+m}{2m}^2 \frac{q^{j^2} - q^{(j+1)^2}}{\eta(q)} . \tag{4.16}$$

It is not clear that this is correct, since we assumed here that the Ishibashi states $|j, m, m\rangle\rangle$ are orthonormal to each other. The zero-mode contribution to the overlap of two such Ishibashi states involves integrals of the form $\int dte^{(n-n')t}$ which is not well-defined. Taking it equal to $\delta_{n,n'}$ leads to (4.16). However, it is not obvious that this the correct thing to do. Another peculiar feature of (4.16) is that it contains binomial coefficients that grow very rapidly. All these problems are related to the fact that (4.6) is an expansion in non-normalizable states, and a general framework to formulate string amplitudes in terms of such states does not yet exist.

The boundary state (4.4) on the other hand is a much more conventional expression in terms of normalizable Ishibashi states. It is this type of expansion that we would normally use when computing string theory amplitudes, and we will use (4.4) in the rest of this section. It turns out that (4.4) yields answers that can be directly compared to matrix model calculations. It remains to be seen whether and how the same results can be recovered from (4.6).

[2]The Kac determinant for $c = 1$ theories at level N is $\Delta_N(h, c) \sim \prod_{pq \leq N} (h - (p-q)^2/4)^{P(N-pq)}$ and clearly this has no zeroes for $h < 0$.

[3]The absence of null-vectors is related to the statement in [21] that one can choose a gauge at nonzero energy where the timelike oscillators are not present.

Except for the half-brane (4.1) we will also be interested in the full-brane (4.2). For the full brane a discussion similar to the one above applies, the main difference being that $\rho(\lambda, t)$ in (4.12) gets replaced by

$$\rho(\lambda, t) = \frac{1}{1+\hat{\lambda}e^{t}} + \frac{1}{1-\hat{\lambda}e^{-t}} - 1\,, \qquad \hat{\lambda} = \sin\lambda\pi\,. \tag{4.17}$$

For the full brane we need to choose in (4.13) an integration contour. One can choose a real contour, which corresponds to a brane building up from a finely tuned tachyon state, and then decaying back. Another contour choice is the Hartle-Hawking contour, consisting of a piece along the imaginary time axis from $t \in (0, i\infty)$, and a real part $t \in (0, \infty)$ (or $t \in (-\infty, 0)$). These correspond to a preparing a state at $t = 0$ and evolving it to (from) the asymptotic infinity [21].

4.3 Cylinder computation

We now concentrate on computing the cylinder diagram of two decaying D0-branes. In terms of boundary states it can be written as

$$A_{\mathrm{ZZ}} = \frac{1}{2}\int_0^1 \frac{dq}{q}\langle B_{ZZ}| \otimes \langle B_{\lambda'}|q^{2L_0}|B_\lambda\rangle \otimes |B_{ZZ}\rangle \mathrm{A}_{\mathrm{ghost}}\,. \tag{4.18}$$

In this expression $|B_\lambda\rangle$ is the time dependent part of the boundary state, and $|B_{ZZ}\rangle$ is the Liouville part. The ghost contribution is separated into A_{ghost}. We consider the overlap of two different rolling boundary states, parametrized by λ and λ'. Also, q is the modulus of the annulus. Since the interacting states are time-dependent, we compute the full time-integrated diagram.

The Liouville part of the annulus diagram is already given in [14, 23],

$$Z_{1,1} = \int dp \chi_p \Psi_{1,1}(p)\Psi_{1,1}(-p) \tag{4.19}$$

where the index $\{1,1\}$ refers to the fact that we take the $m = 1$, $n = 1$ boundary state, $\Psi_{1,1}$ is the wave function, and the character is

$$\begin{aligned} \chi_p &= \frac{q^{p^2}}{\eta(\tau)} \\ \eta(\tau) &= q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) \end{aligned} \tag{4.20}$$

More explicitly

$$Z_{1,1} = -\int \frac{dp}{2\pi i}\frac{q^{p^2}}{\eta(q)}\left(\frac{2}{\pi}\sinh(\pi p)\right)^2. \tag{4.21}$$

The ghost part of the annulus is

$$A_{\mathrm{ghost}} = (\eta(q))^2\,. \tag{4.22}$$

We still need to compute the time-dependent part. Inserting Ishibashi states built over $e^{i\omega X^0}|0\rangle$

$$A_{\lambda,\lambda'} = \langle B_{\lambda'}|q^{2L_0}|B_\lambda\rangle = \int dt \int dt' \int d\omega \int d\omega' \langle B_{\lambda'}|\omega\rangle\langle\!\langle\omega|q^{2L_0}|\omega'\rangle\!\rangle\langle\omega'|B_\lambda\rangle \tag{4.23}$$

$$= \int d\omega \frac{q^{-\omega^2}}{\eta(q)}\rho(\lambda,t)\rho(\lambda',t')e^{i\omega(t-t')}\,. \tag{4.24}$$

As we discussed in the previous subsection, we used for our boundary state one that is expanded purely in terms of normalizable Ishibashi states, and which is therefore determined by the zero-mode structure of the corresponding euclidean boundary state, together with conformal invariance.

Assembling the full annulus amplitude and performing the integration over the modulus q we obtain

$$A_{\rm ZZ} = -\frac{1}{2}\int dt \int dt' \int \frac{dp}{2\pi i}\int d\omega\rho(\lambda,t)\rho(\lambda',t')\frac{(\sinh(\pi p))^2}{p^2-\omega^2}e^{i\omega(t-t')}\,. \tag{4.25}$$

Note that the ghost contribution canceled the η-functions. We now have a choice to proceed with the ω or with the t integration first. Let us proceed with the ω integration,

$$A_{\rm ZZ} = -\int dt \int dt' \int \frac{dp}{p}\rho(\lambda,t)\rho(\lambda',t')\left(\frac{\sinh(\pi p)}{\pi}\right)^2 e^{ip(t-t')}\,. \tag{4.26}$$

Finally for the time integrations the contour has to be specified. The integral of $\rho(t)$ has poles at discrete values on the imaginary axis. For the half-brane we follow the prescription described in the previous section, with the result

$$\int dt\rho(t)e^{ipt} = \frac{-i\pi}{\sinh \pi p}e^{-ip\log\hat{\lambda}}\,. \tag{4.27}$$

So finally we arrive at

$$A_{\rm ZZ} = \int \frac{dp}{p}e^{-ip(\log\hat{\lambda}-\log\hat{\lambda}')}\,. \tag{4.28}$$

This result is in agreement with the matrix model computations 2.13 and 3.39.

If we consider full-brane or bounce solution with real contour, the annulus amplitude is modified to

$$A_{\rm ZZ} = \int \frac{dp}{p}\left(e^{-ip\log\hat{\lambda}} - e^{+ip\log\hat{\lambda}}\right)\left(e^{+ip\log\hat{\lambda}'} - e^{-ip\log\hat{\lambda}'}\right)\,. \tag{4.29}$$

Had we done the t integration first, we would have obtained instead

$$A_{\rm ZZ} = \frac{\pi^2}{2}\int d\omega \frac{dp}{2\pi i}\frac{1}{(p^2-\omega^2)}\left(\frac{\sinh\pi p}{\sinh\pi\omega}\right)^2 e^{-i\omega(\log\hat{\lambda}-\log\hat{\lambda}')}\,. \tag{4.30}$$

Here we have taken the half-brane case for simplicity. Since the p integral diverges, we now have to proceed with the ω integration. The expression has double poles at discrete imaginary values of ω, which indicates the presence of discrete modes. Performing the ω integration we get the previous result (4.28) plus an infinite sum containing the contribution from the discrete modes. This infinite sum with contributions from discrete modes does not seem to be reflected in the matrix model result. The expression (4.28) contains only an exchange of intermediate tachyons, and this result is in agreement with the matrix model computations.

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5. Discussion

We have seen that the matrix model computations performed in the free fermionic picture agree with the string theory cylinder diagram in which only the tachyon exchange is taken into account. The matrix model correlators in the tree level ($g_s \to 0$) approximation do not indicate the presence of discrete modes.

On the other hand, one expects a sign of the presence of discrete states, at least in euclidean signature, where they correspond to normalizable string states. In that case they are part of the full spectrum, and carry important physical information such as gravitational interactions. The discrete modes certainly can be seen to be implicitly present in the matrix model through the ground ring generators and the w_∞ algebra [24].

In the cylinder computation between the two D0-branes we have encountered several difficulties. A better understanding of these difficulties might provide interesting insights into the nature of the matrix model, in particular regarding the role of the discrete modes and its non-perturbative degrees of freedom. In particular, we would like to know whether the cylinder calculation can be reformulated purely in terms of (4.6) or not. Another confusion arises relating a euclidean worldsheet to a Minkowski spacetime computation.

Another inherent difficulty with the cylinder computation is the divergence of the p-integration. We have chosen to take integrals in a particular order to avoid the divergences. This corresponds to regulating the divergent integral in a specific way. We have chosen the regularization to be in agreement with the matrix model computation, but again, this regularization procedure cuts off an infinite sum of discrete modes. Clearly, a better understanding of the string theory computation is desirable.

There are several ways in which the results in this paper can be extended. First of all, we can consider the interactions of $p > 2$ branes. From the matrix model point of view, this can also be understood as taking $(N+p) \times (N+p)$ matrices, integrating out the $N \times N$ degrees of freedom and taking a large N limit. According to [19], the result is given by a Kontsevich matrix integral whose result is a certain τ-function. This τ-function has an alternative representation as a determinant of fermion bilinears, which is precisely what one obtains from a straightforward generalization of (3.35). This provides further evidence for the structure advocated in section 3. It would be interesting to understand more directly how such a structure can emerge from world-sheet calculations.

It is also relatively straightforward to generalize our results to rolling tachyons in the $\hat{c} = 1$ model of [10]. This simply amounts to filling both sides of the potential, and incorporating both asymptotic regions of the matrix model. In particular, we can study rolling tachyons corresponding to perturbations of the form $\lambda \sinh X^0$, which start at early times at $x \to -\infty$ and at late times end up at $x \to +\infty$. It is easy to write down corresponding expressions in the matrix model, but they are more difficult to understand on the world-sheet, as these are configurations that penetrate deeply in the non-perturbative regime; they start out as coherent states of the symmetric combination of the closed string tachyon and RR scalar, and end up as coherent states of the antisymmetric combination instead.

Our calculations also clearly support Sen's point of view [20] that open and closed strings are dual pictures of one and the same thing. The matrix model as expressed in terms of the original variables (eigenvalues) is the open string picture. The fermionic description, which is a reformulation of the original matrix model, represents another formulation. The dual closed string picture arises from bosonization of the fermions. In the closed string picture there is a clear distinction between closed string states and D-branes: the second are coherent states of the first (in other words, they are solitons in the closed string). Similarly, in the open string picture closed strings arise as excitations of brane/antibrane systems with total brane number zero. Thus, both in the closed as well as in the open picture, there is a conserved fermion=brane number, which is conserved despite the fact that the branes are unstable.

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A. Stationary phase limit of the two point function

In this appendix, we consider the full two point correlator of the free fermions without taking any asymptotic bosonization. At the genus zero limit, we can take the stationary phase approximation. This approximation extracts the piece of the correlator corresponding to the tree level ($g_s \to 0$) string computation discussed in section (4).

The full fermionic two point correlator is [18]

$$\begin{aligned}\langle\mu|T(\psi(t_1,x_1)\psi(t_2,x_2)|\mu\rangle^E = & \int d\omega\,(\theta(\omega-\mu)\theta(\Delta t)-\theta(\mu-\omega)\theta(-\Delta t))\,e^{-\omega\Delta t}\times \\ & \times(\psi^+(\omega,x_1)\psi^+(\omega,x_2)+\psi^-(\omega,x_1)\psi^-(\omega,x_2))\,. \end{aligned}\tag{A.1}$$

The subscript E refers to taking the euclidean two point function. Using the properties of wave functions of the inverse harmonic oscillator, or by analytically continuing the harmonic oscillator propagator we arrive at the equivalent expression [18]

$$\begin{aligned}\langle\mu|T(\psi(t_1,x_1)\psi(t_2,x_2)|\mu\rangle^E = & \, ie^{-\mu\Delta t}\int\frac{dp}{2\pi}e^{-ip\Delta t}\int_0^{\mathrm{sgn(p)}\cdot\infty} ds\frac{e^{-sp+i\mu s}}{(-4\pi i\sinh s)^{1/2})}\times \\ & \times\exp\left(-\frac{i\left((x_1^2+x_2^2)\cosh s-2x_1x_2\right)}{4\sinh s}\right). \end{aligned}\tag{A.2}$$

To consider tree level interaction, we suppress the higher genus corrections. We rescale $x\to\beta^{1/2}x$, and $\mu\to\beta\mu$, where β is sent to infinity and $\beta\mu$ is kept fixed but large. Taking this limit suppresses both loop corrections (these are effectively $\hbar = 1/\beta$ corrections), and tree level ($1/\beta\mu$) higher order interactions. The first corrections would correspond to the higher genus corrections in the string diagrams. The tree level corrections would come from taking into account the interactions of the bosonized scalar field. Since in this paper we are mostly interested in the cylinder computation on the string side, we keep $\beta\mu \gg 1$.

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Rescaling x the integral acquires a large phase, and can be evaluated with the stationary phase method. The phase will be stationary at

$$e^{\bar{s}} + e^{-\bar{s}} = \frac{x_1}{x_2} + \frac{x_2}{x_1} \tag{A.3}$$

where $\bar{s}$ denotes the stationary value of s. Depending on the signature of p in the integration and on the signature of $(x_1 - x_2)$, this gives

$$\begin{aligned}\bar{s} = \Theta(p)\left(\Theta(x_1-x_2)\log\left(\frac{x_1}{x_2}\right) + \Theta(x_2-x_1)\log\left(\frac{x_2}{x_1}\right)\right) + \\ +\Theta(-p)\left(\Theta\left(x_1-x_2\right)\log\left(\frac{x_2}{x_1}\right) + \Theta(x_2-x_1)\log\left(\frac{x_1}{x_2}\right)\right).\end{aligned} \tag{A.4}$$

Here Θ is the step function. At the stationary point part of the gaussian factor coming from the integration of the quadratic part cancels the $1/\sqrt{\sinh s}$ factor in the integral. The rest of the expression assembles to the WKB wave function, and an additional exponential factor, giving

$$\begin{aligned}S(t_1,t_2,x_1,x_2)^E = i\sqrt{\frac{2}{\beta|x_1x_2|}}e^{-\beta\mu\Delta t}\int_0^\infty \frac{dp}{2\pi}e^{-ip\Delta t}\times \\ \times\Bigg\{\Theta(p)\Bigg[\Theta(x_1-x_2)e^{-\frac{i\beta}{4}(x_1^2-x_2^2)}e^{(i\beta\mu-p)\log\frac{x_1}{x_2}} + \\ +\Theta(x_2-x_1)e^{-\frac{i\beta}{4}(x_2^2-x_1^2)}e^{(i\beta\mu-p)\log\frac{x_2}{x_1}}\Bigg] + \\ +\Theta(-p)\Bigg[\Theta(x_1-x_2)e^{-\frac{i\beta}{4}(x_2^2-x_1^2)}e^{(i\beta\mu-p)\log\frac{x_2}{x_1}} + \\ +\Theta(x_2-x_1)e^{-\frac{i\beta}{4}(x_1^2-x_2^2)}e^{(i\beta\mu-p)\log\frac{x_1}{x_2}}\Bigg]\Bigg\}.\end{aligned} \tag{A.5}$$

Going back to Minkowski signature requires $\Delta t \to i\Delta t$, $p \to ip$. The four pieces of the integral correspond to picking the left or right moving pieces in the correlator: $\Psi^\dagger_{L,R}\Psi_{L,R}$. Dropping the WKB factor and choosing the left-left piece we obtain for the two point correlator

$$S(t_1,t_2,x_1,x_2) = \int_0^\infty dp e^{-ip\left(\log\frac{x_1}{x_2}-\Delta t\right)}. \tag{A.6}$$

Taking the orbits $x_{1,2} = \hat{\lambda}_{1,2}\cosh t$ (where t is sent to infinity) we obtain the result (2.13) computed from the asymptotic bosonization.

Notice that to get this result, we have only suppressed higher genus and higher order tree level correction, we did not consciously remove any possible discrete modes.

B. Two point function of eigenvalue density

In this appendix we consider two point function of the eigenvalue density. Roughly speaking, since for chiral fermions $\psi(x,t) \sim \exp\left(\int\psi^\dagger\psi(x,t)\right)$, by computing the two point function of the eigenvalue density we can try to see once more whether discrete modes are exchanged between two D-branes, and we can also present an alternative derivation of the cylinder diagram.

The two point function of eigenvalue density is defined in terms of

$$\begin{aligned} G(t_1,x_1;t_2,x_2) &= \langle\mu|\psi^\dagger\psi(t_1,x_1)\psi^\dagger\psi(t_2,x_2)|\mu\rangle \\ G(\omega_1,x_1;\omega_2,x_2) &= \int dt_1 dt_2 e^{i\omega_1 t_1} e^{i\omega_2 t_2} G(t_1,x_1;t_2,x_2)\,. \end{aligned} \tag{B.1}$$

We can compute this using the two point functions of fermions computed in appendix A,

$$G(\omega_1,x_1;\omega_2,x_2) = i^2\delta(\omega_1+\omega_2)\int_0^\infty ds\int_{-\infty}^\infty dt\frac{1}{s+t}e^{-s\omega_1}e^{i\mu(s+t)}\langle x_1|e^{2isH}|x_2\rangle\langle x_2|e^{2itH}|x_1\rangle\,, \tag{B.2}$$

where $\langle x_m|e^{2isH}|x_n\rangle$ defined as

$$\langle x_m|e^{2isH}|x_n\rangle = \frac{1}{(-4\pi i\sinh s)^{1/2}}\exp\left(-\frac{i\left((x_m^2+x_n^2)\cosh s - 2x_m x_n\right)}{4\sinh s}\right). \tag{B.3}$$

Here we have taken the euclidean time. As in the appendix A, We will think of the genus zero limit, scaling $\mu\to\beta\mu, x\to\sqrt{\beta}x$ and taking the limit $\beta\to\infty$ with $\beta\mu=$ fixed $\gg 1$. Then we can apply the stationary phase approximation to the s and t integration. But as before, there are two sets of saddle points: $\bar{s}=\bar{t}=\pm\log(x_1/x_2)$. To have a non-vanishing denominator in (B.2), we choose the same sign for the saddle of $\bar{s}$ and $\bar{t}$. Finally, we get

$$G(\omega_1,x_1;\omega_2,x_2) = \delta(\omega_1+\omega_2)\frac{1}{2x_1x_2\log(x_2/x_1)}e^{(2i\mu-\omega_1)\log(x_2/x_1)-\frac{i}{2}(x_2^2-x_1^2)}\,. \tag{B.4}$$

We take the inverse Fourier transformation to (t,x) coordinate space

$$G(t_1,x_1;t_2,x_2) = \int d\omega_1 d\omega_2 e^{-i\omega_1 t_1}e^{-i\omega_2 t_2}G(\omega_1,x_1;\omega_2,x_2)\,. \tag{B.5}$$

Stripping off WKB wavefunctions, and Wick rotating to Minkowski space, we get

$$\int d\omega\frac{1}{\log x_1-\log x_2}e^{-i\omega(t_1-\log x_1-t_2+\log x_2)}\,. \tag{B.6}$$

To get the cylinder diagram between two D-branes, we should integrate over t in (B.6) up to a large nonzero value of t, and then insert the classical orbit at the boundary value of t, which yields the simple answer

$$\mathcal{A} = \int_0^\infty\frac{d\omega}{\omega}e^{-i\omega\log(\hat{\lambda}_1/\hat{\lambda}_2)}\,. \tag{B.7}$$

Since $\psi(x_1,t_1)\psi(x_2,t_2)\sim\exp\left(\int\psi^\dagger\psi(x_x,t_1)\right)\exp\left(\int\psi^\dagger\psi(x_2,t_2)\right)$, by summing the all the ladders, we recover (2.13). Note that we again see no divergences due to discrete mode exchange (or any other sign of discrete modes for that matter).

We can evaluate the two point function of the eigenvalue density in another way. If we take the Fourier transformation from x to z, and rotate z to il, we get the two point function of macroscopic loop operators of loop length l. The result is already well-known in the literature [29, 18]. At genus zero the result is

$$\langle W(w_1,l_1)W(w_2,l_2)\rangle = \delta(\omega_1+\omega_2)\int\frac{d\zeta}{\pi^2}\frac{\zeta^2}{\zeta^2+\omega_1^2}K_{i\zeta}(-2\sqrt{\mu}l_1)K_{i\zeta}(-2\sqrt{\mu}l_2)\,. \tag{B.8}$$

We can now analytically continue back to $z = -il$ and do an inverse Fourier transformation to get the following expression in terms of x,

$$\frac{\pi}{\sqrt{x_1^2 - 4\mu}\sqrt{x_2^2 - 4\mu}} \int d\omega_1 \omega_1 e^{-i\omega_1(t_1 - t_2)} \times$$
$$\times \cosh\left(2\omega_1 \sinh^{-1} \frac{\sqrt{\frac{x_1}{\sqrt{\mu}} - 2}}{2}\right) \cosh\left(2\omega_1 \sinh^{-1} \frac{\sqrt{\frac{x_2}{\sqrt{\mu}} - 2}}{2}\right). \tag{B.9}$$

To get the Minkowski result we perform a Wick rotation $t \to -it$ and $\omega \to i\omega$. Again taking off the WKB factors, we have

$$\sim \int d\omega_1 \omega_1 e^{-i\omega_1(t_1 - t_2 - \log x_1 + \log x_2)}. \tag{B.10}$$

Also here we should do the integral over t to reproduce the correct answer for the cylinder diagram,

$$\begin{aligned} \mathcal{A} &= \int_0^\infty dt_1 dt_2 \int d\omega_1 \omega_1 e^{-i\omega_1(t_1 - t_2 - \log x_1 + \log x_2)} \\ &= \int \frac{d\omega}{\omega} e^{-i\omega \log(\hat{\lambda}_1/\hat{\lambda}_2)}. \end{aligned} \tag{B.11}$$

In the last line, we have again inserted the classical trajectory at asymptotic infinity. We find that the result agrees completely with the previous results.

References

[1] P. Ginsparg and G.W. Moore, *Lectures on 2-D gravity and 2-D string theory*, hep-th/9304011.

[2] I.R. Klebanov, *String theory in two-dimensions*, hep-th/9108019.

[3] J. Polchinski, *What is string theory?*, hep-th/9411028.

[4] J. McGreevy and H. Verlinde, *Strings from tachyons: the $c = 1$ matrix reloaded*, *J. High Energy Phys.* **12** (2003) 054 [hep-th/0304224].

[5] J. McGreevy, J. Teschner and H. Verlinde, *Classical and quantum D-branes in 2D string theory*, *J. High Energy Phys.* **01** (2004) 039 [hep-th/0305194].

[6] I.R. Klebanov, J. Maldacena and N. Seiberg, *D-brane decay in two-dimensional string theory*, *J. High Energy Phys.* **07** (2003) 045 [hep-th/0305159].

[7] S.Y. Alexandrov, V.A. Kazakov and D. Kutasov, *Non-perturbative effects in matrix models and D-branes*, *J. High Energy Phys.* **09** (2003) 057 [hep-th/0306177];
D. Gaiotto, N. Itzhaki and L. Rastelli, *On the bCFT description of holes in the $c = 1$ matrix model*, *Phys. Lett.* **B 575** (2003) 111 [hep-th/0307221];
J. McGreevy, S. Murthy and H. Verlinde, *Two-dimensional superstrings and the supersymmetric matrix model*, hep-th/0308105;
J.L. Karczmarek and A. Strominger, *Matrix cosmology*, hep-th/0309138;
I.R. Klebanov, J. Maldacena and N. Seiberg, *Unitary and complex matrix models as 1-d type 0 strings*, hep-th/0309168;

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S. Dasgupta and T. Dasgupta, *Renormalization group approach to $c = 1$ matrix model on a circle and D-brane decay*, hep-th/0310106;
S. Alexandrov, (m, n) *ZZ branes and the $c = 1$ matrix model*, hep-th/0310135;
J. Gomis and A. Kapustin, *Two-dimensional unoriented strings and matrix models*, hep-th/0310195;
S. Mukhi, *Topological matrix models, Liouville matrix model and $c = 1$ string theory*, hep-th/0310287;
O. Bergman and S. Hirano, *The cap in the hat: unoriented 2d strings and matrix(- vector) models*, *J. High Energy Phys.* **01** (2004) 043 [hep-th/0311068];
D.J. Gross and J. Walcher, *Non-perturbative RR potentials in the $\hat{c} = 1$ matrix model*, hep-th/0312021.

[8] N.R. Constable and F. Larsen, *The rolling tachyon as a matrix model*, *J. High Energy Phys.* **06** (2003) 017 [hep-th/0305177].

[9] M. Gutperle and P. Kraus, *D-brane dynamics in the $c = 1$ matrix model*, hep-th/0308047.

[10] M.R. Douglas et al., *A new hat for the $c = 1$ matrix model*, hep-th/0307195.

[11] T. Takayanagi and N. Toumbas, *A matrix model dual of type 0b string theory in two dimensions*, *J. High Energy Phys.* **07** (2003) 064 [hep-th/0307083].

[12] O. DeWolfe, R. Roiban, M. Spradlin, A. Volovich and J. Walcher, *On the S-matrix of type 0 string theory*, *J. High Energy Phys.* **11** (2003) 012 [hep-th/0309148].

[13] V. Fateev, A.B. Zamolodchikov and A.B. Zamolodchikov, *Boundary Liouville field theory, I. Boundary state and boundary two-point function*, hep-th/0001012.

[14] A.B. Zamolodchikov and A.B. Zamolodchikov, *Liouville field theory on a pseudosphere*, hep-th/0101152.

[15] A. Sen, *Rolling tachyon*, *J. High Energy Phys.* **04** (2002) 048 [hep-th/0203211]; *Tachyon matter*, *J. High Energy Phys.* **07** (2002) 065 [hep-th/0203265]; *Time evolution in open string theory*, *J. High Energy Phys.* **10** (2002) 003 [hep-th/0207105].

[16] D.J. Gross and I.R. Klebanov, *$S = 1$ for $c = 1$*, *Nucl. Phys.* **B 359** (1991) 3.

[17] G.W. Moore, M.R. Plesser and S. Ramgoolam, *Exact s matrix for 2-d string theory*, *Nucl. Phys.* **B 377** (1992) 143 [hep-th/9111035].

[18] G.W. Moore, *Double scaled field theory at $c = 1$*, *Nucl. Phys.* **B 368** (1992) 557.

[19] R. Dijkgraaf, G.W. Moore and R. Plesser, *The partition function of 2-D string theory*, *Nucl. Phys.* **B 394** (1993) 356 [hep-th/9208031].

[20] A. Sen, *Open-closed duality: lessons from matrix model*, hep-th/0308068.

[21] N. Lambert, H. Liu and J. Maldacena, *Closed strings from decaying D-branes*, hep-th/0303139;
J.L. Karczmarek, H. Liu, J. Maldacena and A. Strominger, *UV finite brane decay*, *J. High Energy Phys.* **11** (2003) 042 [hep-th/0306132].

[22] M. Gutperle and A. Strominger, *Timelike boundary Liouville theory*, *Phys. Rev.* **D 67** (2003) 126002 [hep-th/0301038].

[23] E.J. Martinec, *The annular report on non-critical string theory*, hep-th/0305148.

[24] E. Witten, *Ground ring of two-dimensional string theory*, *Nucl. Phys.* **B 373** (1992) 187 [hep-th/9108004].

[25] C.G. Callan Jr., I.R. Klebanov, A.W.W. Ludwig and J.M. Maldacena, *Exact solution of a boundary conformal field theory*, *Nucl. Phys.* **B 422** (1994) 417 [hep-th/9402113].

[26] M.R. Gaberdiel, A. Recknagel and G.M.T. Watts, *The conformal boundary states for* SU(2) *at level 1*, *Nucl. Phys.* **B 626** (2002) 344 [hep-th/0108102].

[27] K. Okuyama, *Comments on half S-branes*, *J. High Energy Phys.* **09** (2003) 053 [hep-th/0308172].

[28] F. Larsen, A. Naqvi and S. Terashima, *Rolling tachyons and decaying branes*, *J. High Energy Phys.* **02** (2003) 039 [hep-th/0212248].

[29] T. Banks, M.R. Douglas, N. Seiberg and S.H. Shenker, *Microscopic and macroscopic loops in nonperturbative two-dimensional gravity*, *Phys. Lett.* **B 238** (1990) 279.